Review of Some Approximation Operators for the Numerical Analysis of Spectral Methods

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Abstract. This paper reviews some operators that are used in the numerical analysis of spectral and spectral element methods. We motivate the introduction of these different operators and sketch their approximation properties. Finally, we apply them to derive optimal error estimates for spectral type approximations of the solution of elliptic partial differential equations.

§1. Introduction

Spectral type methods are high order discretizations that allow to compute approximate solutions of partial differential equations. The recent version of spectral approximations is based on the Galerkin approach where the variational statement (equivalent to the strong formulation of the PDE) is set on discrete spaces of test and trial functions. For instance, let us consider the problem: *find* $u \in X$ *such that*

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in X, \tag{1}$$

where X is some Hilbert space, and a is a continuous bilinear form over X. The general Galerkin approximation of this problem first requires the choice of a family of discrete spaces $X_N \subset X$, where N is a parameter that tends to infinity and is related to the dimension of the discrete space X_N. The discrete problem is then stated as follows: *find* $u_N \in X_N$ *such that*

$$a(u_N, v_N) = \langle f, v_N\rangle, \quad \forall v_N \in X_N. \tag{2}$$

The basic general hypothesis that makes problem (1) well-posed is that a is continuous and α-elliptic over X (i.e. $\exists\ \alpha > 0$ such that $a(u,u) \geq \alpha\|u\|_X^2$ for

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all $u \in X$. These properties remain true over each X_N (since $X_N \subset X$); thus (2) is also well-posed for each N. In addition, the solution u_N satisfies

$$\|u - u_N\|_X \leq c \inf_{v_N \in X_N} \|u - v_N\|_X. \tag{3}$$

The constant c that appears in (3) is the quotient of the continuity constant of a with the ellipticipty constant α and is thus independent of X_N.

Going back to spectral methods, the definition of X_N involves polynomials, and in the most simple cases (we shall see more general examples in Section 5) we have $X_N = X \cap \mathbb{P}_N$, where $\mathbb{P}_N$ represents the set of all polynomials of (partial) degree less than or equal to N. Here N is the parameter responsible for the convergence of the method. Due to (3), one ingredient in the numerical analysis of the spectral method is the approximation properties of the space of polynomials for given functions. The classical analysis of the approximation properties of polynomials is done in terms of L^∞–norms. This is not completely appropriate for our purpose since most often X is a Hilbert space (generally L^2 or H^1 spaces), and the approximation properties have to be measured with these norms. If a rate of convergence (with respect to N) on the best fit $\inf_{v_N \in X_N} \|u - v_N\|_X$ is sought after, some regularity has to be assumed over u. In Section 2, we give a survey of these best approximation results depending on the regularity of the function we want to approximate. We first analyze the L^2–best fit and then the H^1–best fit. The main ingredient in this analysis relies on the Legendre basis that is composed of the orthogonal polynomials for the standard Lebesgue mesure over the interval $(-1, +1)$. These polynomials, denoted as $(L_n)_n$, are defined by: degree$(L_n) = n$,

$$L_n(1) = 1, \tag{4}$$

$$\int_{-1}^{1} L_n(\zeta) L_m(\zeta) d\zeta = \frac{2\delta_{m,n}}{2n+1}. \tag{5}$$

They satisfy some standard properties (actually valid for most families of orthogonal polynomials)

$$\mathcal{A}(L_n) \equiv -\frac{d}{d\zeta}((1-\zeta^2)\frac{dL_n}{d\zeta} = n(n+1)L_n, \tag{6}$$

that one can translate by saying that the Legendre polynomials are the eigenvectors of the (Sturm-Liouville) operator $\mathcal{A}$. Since this is a possible basis set for the implementation of problem (2), this gives the name of spectral to the methods we shall consider hereafter, and that have been first analyzed in [10]. We refer also to [6] and [3] for more recent surveys on the numerical analysis of these methods. In Section 3, we introduce the notion of numerical integration and the interpolation operator, two notions that are naturally quite close and that allow to transform the "theoretical" approximation method into a "applicable" one. In Section 4, motivated by the analysis of the Stokes problem, we introduce a new operator, that, in opposition to the previous ones, is uniformly stable (in N) both in the L^2–norm and the H^1–norm and possess optimal approximation properties. It has to be said, beforehand, that in the precise analysis of these spectral (or polynomial) approximation, the Bernstein inequality runs counter to most standard tools that generally allow for deriving approximation results for a new operator from an already analyzed one. This Bernstein inequality tells about the equivalence of norms on the finite dimensional linear space of polynomials. It is well known that, for any function in H^1, the L^2 norm is smaller than the H^1–norm; of course this is true in particular on polynomials:

$$\forall \phi_N \in \mathbb{P}_N, \qquad \|\phi_N\|_{L^2} \leq \|\phi_N\|_{H^1}.$$

Since all norms are equivalent on $\mathbb{P}_N$, there exists a constant (obviously depending on N) such that

$$\forall \phi_N \in \mathbb{P}_N, \qquad \|\phi_N\|_{H^1} \leq c(N)\|\phi_N\|_{L^2}.$$

The behaviour of this constant is made precise by the Bernstein inequality

$$\forall \phi_N \in \mathbb{P}_N, \qquad \|\phi_N\|_{H^1} \leq cN^2\|\phi_N\|_{L^2},$$

where c no longer depends on N. This estimate is optimal (in the sense that there exists a sequence of polynomials such that the ratio of the H^1–norm over the L^2–norm scales like $\mathcal{O}(N^2)$), but is bad as regards the ratio of convergence rate between the H^1–best fit and the L^2–best fit that scales like $\mathcal{O}(N^{-1})$, as we shall see below.

In the first three sections, the domains where the functions live will be very simple, actually too simple to tackle real life problems; indeed these are bricks equal to $(-1,1)^d$ where $d = 1, 2$ or 3. The generalization of spectral methods to more complex geometries is done by combining two key ingredients: the mapping of bricks onto curved bricks through regular mappings, and domain decomposition. We give some hints about this generalization in §5.

§2. Hilbert Type Projection Operators

Let us start with the one-dimensional case. In $L^2(-1,1)$, we consider the set $\mathbb{P}_N(-1,1)$ of all polynomials of degree $\leq N$. From the Weierstrass density theorem, we know that any element ϕ in $L^2(-1,1)$ can be written as

$$\phi(\zeta) = \sum_{n=0}^{\infty} \widehat{\phi}^n L_n(\zeta), \tag{7}$$

where the convergence of the series holds in L^2. The coefficients $\widehat{\phi}^n$ can be derived from ϕ thanks to the orthogonality of the Legendre basis as follows:

$$\widehat{\phi}^n = \frac{2n+1}{2} \int_{-1}^{1} \phi(\zeta) L_n(\zeta) d\zeta.$$

Next, from (6) we derive that

$$\widehat{\phi}^n = \frac{2n+1}{2}\int_{-1}^{1}\phi(\zeta)\frac{\mathcal{A}L_n(\zeta)}{n(n+1)}d\zeta,$$

noticing that $\mathcal{A}$ is symmetric, and assuming ϕ regular enough, we derive that

$$\widehat{\phi}^n = \frac{2n+1}{2}\int_{-1}^{1}\mathcal{A}(\phi)(\zeta)\frac{L_n(\zeta)}{n(n+1)}d\zeta.$$

If we iterate this argument p times, we obtain

$$\widehat{\phi}^n = \frac{2n+1}{2}\int_{-1}^{1}\mathcal{A}^p(\phi)(\zeta)\frac{L_n(\zeta)}{n^p(n+1)^p}d\zeta,$$

so that, the following simple relation holds between the Legendre coefficients of ϕ and of $\mathcal{A}^p(\phi)$:

$$\widehat{\phi}^n = \frac{1}{n^p(n+1)^p}\widehat{\mathcal{A}^p(\phi)}^n.$$

Next, let π_N denote the $L^2(-1,1)$–projection over $\mathbb{P}_N(-1,1)$. Going back to (6), we deduce from (7) and (5) that

$$\pi_N(\phi) = \sum_{n=0}^{N}\widehat{\phi}^n L_n(\zeta), \tag{8}$$

so that

$$\phi - \pi_N(\phi) = \sum_{n=N+1}^{\infty}\widehat{\phi}^n L_n(\zeta) = \sum_{n=N+1}^{\infty}\frac{1}{n^p(n+1)^p}\widehat{\mathcal{A}^p(\phi)}^n L_n(\zeta)$$

and, by Parseval

$$\begin{aligned}\|\phi - \pi_N(\phi)\|^2_{L^2(-1,1)} &= \sum_{n=N+1}^{\infty}[\frac{1}{n^p(n+1)^p}]^2[\widehat{\mathcal{A}^p(\phi)}^n]^2\frac{2}{2n+1}\\ &\le [\frac{1}{N}]^{4p}\sum_{n=N+1}^{\infty}[\widehat{\mathcal{A}^p(\phi)}^n]^2\frac{2}{2n+1}\\ &\le [\frac{1}{N}]^{4p}\sum_{n=0}^{\infty}[\widehat{\mathcal{A}^p(\phi)}^n]^2\frac{2}{2n+1} = [\frac{1}{N}]^{4p}\|\mathcal{A}^p(\phi)\|^2_{L^2(-1,1)}.\end{aligned}$$

We have thus proven that, for any ϕ in the domain $\mathcal{D}[\mathcal{A}^p]$ of $\mathcal{A}^p$,

$$\|\phi - \pi_N(\phi)\|_{L^2(-1,1)} \le c(p)N^{-2p}\|\mathcal{A}^p(\phi)\|_{L^2(-1,1)}.$$

It is easy to check that $H^{2p}(-1,1) \subset \mathcal{D}[\mathcal{A}^p]$; hence the following theorem (due to Canuto and Quarteroni [7]), proven here for even values of r, holds for any r thanks to an argument of interpolation between Sobolev spaces:

Theorem 1. *For any real number $r \geq 0$, there exists a constant $c > 0$, depending only on r such that, for any function $\phi \in H^r(-1,1)$,*

$$\|\phi - \pi_N(\phi)\|_{L^2(-1,1)} \leq cN^{-r}\|\phi\|_{H^r(-1,1)}. \tag{9}$$

Let us denote now by Π_N the $L^2((-1,1)^d)$ orthogonal projection operator over the set $\mathbb{P}_N((-1,1)^d)$ of all polynomials of degree $\leq N$ with respect to each variable. By Fubini's theorem, $\Pi_N = \pi_N \otimes \pi_N$, in 2D and $\Pi_N = \pi_N \otimes o\pi_N \otimes o\pi_N$, in 3D. By tensorizing (9) we derive

Theorem 2. *For any real number $r \geq 0$, there exists a constant $c > 0$, depending only on r such that, for any function $\phi \in H^r((-1,1)^d)$,*

$$\|\phi - \Pi_N(\phi)\|_{L^2((-1,1)^d)} \leq cN^{-r}\|\phi\|_{H^r((-1,1)^d)}.$$

We are now in a position to tackle the approximation in the H^1 norms. First, we consider a function $\phi \in H_0^1(-1,1) \cap H^r(-1,1)$, with $r \geq 1$. It is quite immediate to check that the polynomial $\phi_N(\zeta) = \int_{-1}^{\zeta} \pi_{N-1}\frac{d\phi}{d\xi}(\xi)d\xi$ belongs to $\mathbb{P}_N(-1,1)$, vanishes at $\zeta = -1$, and satisfies

$$\begin{aligned}
\phi_N(1) &= \int_{-1}^{1} \pi_{N-1}\frac{d\phi}{d\xi}(\xi)d\xi = \int_{-1}^{1} \pi_{N-1}\frac{d\phi}{d\xi}(\xi)L_0(\xi)d\xi \\
&= \int_{-1}^{1} \frac{d\phi}{d\xi}(\xi)L_0(\xi)d\xi = \int_{-1}^{1} \frac{d\phi}{d\xi}(\xi)d\xi \\
&= \phi(1) - \phi(-1) = 0,
\end{aligned}$$

and hence is an element of $\mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$. Finally it is a good approximation of ϕ, since from Poincarré's inequality and Theorem 1,

$$\begin{aligned}
\|\phi - \phi_N\|_{H^1(-1,1)} &\leq c\|\frac{d\phi}{d\xi} - \frac{d\phi_N}{d\xi}\|_{L^2(-1,1)} \\
&\leq c\|\frac{d\phi}{d\xi} - \pi_{N-1}(\frac{d\phi}{d\xi})\|_{L^2(-1,1)} \\
&\leq cN^{1-r}\|\frac{d\phi}{d\xi}\|_{H^{r-1}(-1,1)} \leq cN^{1-r}\|\phi\|_{H^r(-1,1)}.
\end{aligned}$$

Let us introduce now the orthogonal projection operator $\pi_N^{1,0}$ from $H_0^1(-1,1)$ onto $\mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$, we can state the following result (due to Maday and Quarteroni [15]):

Theorem 3. *For any real number $r \geq 1$ and any real number $0 \leq s \leq 1$, there exists a constant $c > 0$, depending only on r and s such that for any function $\phi \in H_0^1(-1,1) \cap H^r(-1,1)$,*

$$\|\phi - \pi_N^{1,0}(\phi)\|_{H^s(-1,1)} \leq cN^{s-r}\|\phi\|_{H^r(-1,1)}. \tag{10}$$

Proof: The theorem has been obtained for $s = 1$. For $s = 0$ it is obtained through a standard Aubin-Nitsche duality argument, and then for any s by interpolation between Sobolev spaces. □

Remark. At this point it has to be said that the L^2 projection operator π_N does not have optimal approximation properties in the H^1–norm, the only (non-improvable) property that can be obtained is

$$\|\phi - \pi_N(\phi)\|_{H^1(-1,1)} \le cN^{\frac{3}{2}-r}\|\phi\|_{H^r(-1,1)}.$$

We refer to [3] for details and counter-examples.

Remark. It may also be interesting to note that, despite their definition, the previous operators have stability properties in various norms. First for the L^2-operator, we have

$$\|\pi_N\phi\|_{H^1(-1,1)} \le cN^{\frac{1}{2}}\|\phi\|_{H^1(-1,1)},$$

which is related to what we have indicated in the previous remark, but also

$$\|\pi_N^{1,0}\phi\|_{L^2(-1,1)} \le cN^{\frac{1}{2}}\|\phi\|_{L^2(-1,1)},$$

which is rather suprising since, from this (non-uniform) stability property, the H_0^1–projection operator can be extended to (irregular) functions of L^2!!

Again by tensorization of the results of the one dimensional Theorem 3, we exhibit a polynomial that approximates regular functions in $H_0^1((-1,1)^d)$ well, from which we derive approximation properties on the multidimensional projection operator $\Pi_N^{1,0}$ from $H_0^1((-1,1)^d)$ over $\mathbb{P}_N((-1,1)^d) \cap H_0^1((-1,1)^d)$ both in H^1–norm and in L^2–norm (derived by duality):

Theorem 4. *For any real number $r \ge 1$ and any real number $0 \le s \le 1$, there exists a constant $c > 0$, depending only on r and s such that, for any function $\phi \in H_0^1((-1,1)^d) \cap H^r((-1,1)^d)$,*

$$\|\phi - \Pi_N^{1,0}(\phi)\|_{H^s((-1,1)^d)} \le cN^{s-r}\|\phi\|_{H^r((-1,1)^d)}. \tag{11}$$

These results can be completed in order to derive a whole scale of approximation projectors in higher order norms. These are required, e.g. for the analysis of the approximation of fourth-order problems. The general result, concerning the orthogonal projection operator $\Pi_{N,0}^{\rho,\sigma}$ from $H^\rho((-1,1)^d \cap H_0^\sigma((-1,1)^d$ onto $\mathbb{P}_N((-1,1)^d) \cap H_0^\sigma((-1,1)^d)$ is given in the following theorem (due to Maday [11] in 1D, see also [3] for the extension to 2 and 3D):

Theorem 5. *For any real number $0 \le \sigma \le \rho$ and any $0 \le s \le \rho \le r$, there exists a constant $c > 0$, depending only on r, s, ρ, σ such that, for any function $\phi \in H_0^\sigma((-1,1)^d) \cap H^r((-1,1)^d)$,*

$$\|\phi - \Pi_{N,0}^{\rho,\sigma}(\phi)\|_{H^s((-1,1)^d)} \le cN^{s-r}\|\phi\|_{H^r((-1,1)^d)}.$$

Remark. A final remark on these operators is that improved-approximation results in negative norms are also true, and can be obtained in a classical way, by further refering to the Aubin-Nitsche duality argument. Hence, Theorem 5 is also valid for negative values of s.

These results allow us to prove that the approximation of most elliptic variational problems by spectral methods is optimal. As an example, let us consider the (non-constant) Laplace problem on a cube $\Omega = (-1,1)^3$: given a 3×3 matrix, symmetric and uniformly positive definite, we consider the problem of finding $u \in H_0^1(\Omega)$ such that

$$-\text{div}[A\mathbf{grad}]u = f. \tag{12}$$

The approximation then consists in finding an element u_N in $X_N \equiv \mathbb{P}_N(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega A\nabla u_N \nabla v_N = \int_\Omega f v_N, \quad \forall v_N \in X_N. \tag{13}$$

Assuming that $u \in H^r(\Omega)$, we deduce from (3) and (11) that

$$\|u - u_N\|_{H^1(\Omega)} \leq cN^{1-r}\|u\|_{H^r(\Omega)}.$$

As hinted in the introduction, this problem is numerically intractable; indeed the implementation of (13) requires the computation of the two integrals appearing on the left- and the right-hand sides of this equation. The exact computation is most often impossible, and certainly numerically not fast enough. The use of numerical integration rules is the cure to this problem, but in order to combine efficiency and precision, following Gottlieb [9] and Mercier [17], we refer to the use of Gauss type quadrature rule. Indeed, they are well known to be well suited for the integration of polynomials.

§3. Interpolation Operators

Between the different numerical quadrature rules over $(-1,1)$, well suited for polynomial integration, we shall quote here the Legendre-Gauss and Legendre-Gauss-Lobatto ones. We refer to [2] for more details. For the sake of completeness, we recall the definition of these formulae:

Theorem 6. *(Gauss formula) For any real number n, there exists a unique set of points $-1 < \zeta_1^n < \zeta_2^n < \cdots < \zeta_n^n < 1$, and a unique set of positive weights ω_i^n such that for any polynomial $\phi \in \mathbb{P}_{2n-1}(-1,1)$, the following equality holds:*

$$\int_{-1}^1 \phi(\zeta)d\zeta = \sum_{i=1}^n \phi(\zeta_i^n)\omega_i^n.$$

Theorem 7. *(Gauss-Lobatto formula) For any real number n, there exists a unique set of points $-1 = \xi_0^n < \xi_1^n < \cdots < \xi_n^n = 1$, and a unique set of positive weights ρ_i^n such that for any polynomial $\phi \in \mathbb{P}_{2n-1}(-1,1)$, the following equality holds:*

$$\int_{-1}^{1} \phi(\zeta)d\zeta = \sum_{i=0}^{n} \phi(\xi_i^n)\rho_i^n.$$

From now on, we shall assume that the degree of the polynomials for the approximation is fixed to be N, and we shall use $N+1$ points either of Gauss or Gauss-Lobatto type. For the sake of simplicity, these points will be denoted with no superscript, i.e. in all of what follows, we set $\zeta_i \equiv \zeta_i^{N+1}$ and $\xi_i \equiv \xi_i^N$. We recall that these points are the roots (resp. the extrema) of the Legendre polynomials; more precisely, we have

$$\forall i, \quad L_{N+1}(\zeta_i) = 0, \quad (\text{resp. } (1-\xi_i^2)L'_{N+1}(\xi_i) = 0).$$

After tensorization, these one dimensional quadrature rules easily provide quadrature rules on the square and on the cube defined as follows (e.g. in 2D for the Gauss Lobatto formula):

$$\sum_{\mathrm{GL}} \phi \equiv \sum_{i=0}^{N}\sum_{j=0}^{N} \phi(\xi_i, \xi_j)\omega_i\omega_j.$$

The problem that is actually implemented is then the following: find an element u_N in X_N such that

$$\sum_{\mathrm{GL}} A\nabla u_N \nabla v_N = \sum_{\mathrm{GL}} f v_N, \quad \forall v_N \in X_N. \tag{14}$$

Even in the case where A is constant, at least in more than one dimension, the left-hand side is not exactly computed. The problem is no longer of the form (1), and the abstract theory has to be generalized in order to handle this problem as well.

Here is not the place to detail this generalization (see [3], where the complete analysis is performed) but it is natural that the α-ellipticity of the bilinear form on the left-hand side of (14) is again one of the key ingredients and has to be satisfied. This follows from the property, proven in [7]

$$\forall \phi_N \in \mathbb{P}_N(-1,1), \quad \sum_{\mathrm{GL}} \phi_N^2 \geq \int_{-1}^{1} \phi_N^2(\zeta)d\zeta.$$

From this property it can be easily derived that the solution u_N to (14) exists and is unique.

The approximation properties of the polynomial interpolation operator over the Gauss-Lobatto nodes is of great importance in the error bounds. Let i_N denote this operator in one dimension:

$$\forall \phi \in C^0([-1,1]), \quad i_N(\phi) \in \mathbb{P}_N(-1,1) \text{ and } \forall i, 0 \leq i \leq N, \quad i_N(\phi)(\xi_i) = \phi(\xi_i)$$

and let us tensorize it in order to get a two (resp. a three) dimensional operator $\mathcal{I}_N \equiv i_n \otimes i_n$ (resp. $\mathcal{I}_N \equiv i_n \otimes i_n \otimes i_n$). The properties of this operator have been established in [12] and [2], and read as follows:

Theorem 8. *For any real numbers s and r satisfying $r > (d+s)/2$ and $0 \le s \le 1$, there exists a positive constant c depending only on r such that for any ϕ in $H^r((-1,1)^d)$ the following estimate holds*

$$\|\phi - \mathcal{I}_N(\phi)\|_{H^s((-1,1)^d)} \le cN^{s-r}\|\phi\|_{H^r((-1,1)^d)}. \tag{15}$$

It has to be noticed that this operator requires more regularity than the L^2 projection operator, but it is optimal both in the L^2 and the H^1 norms. It has also to be recalled that in the classical approximation properties in the L^∞ norm, the Lebesgue constant appears as a pollution of the approximation properties of the interpolation operator as regards the optimality provided by the corresponding best fit. This is not the case in the L^2–norm. In this direction what we have more precisely is that, for any function ϕ in $H_0^1(-1,1)$,

$$\|i_N\phi\|_{L^2(-1,1)} \le c(\|\phi\|_{L^2(-1,1)} + \frac{1}{N}\|\frac{d\phi}{d\zeta}\|_{L^2(-1,1)}),$$

and for any function ϕ in $H^1(-1,1)$,

$$\|i_N\phi\|_{H^1(-1,1)} \le c\|\phi\|_{H^1(-1,1)}.$$

Another nice property of this operator, that has some importance for nonlinear PDE's, is the following result: for any polynomial $\phi_M \in \mathbb{P}_M(-1,1)$,

$$\|i_N\phi_M\|_{L^2(-1,1)} \le c(1+\frac{M}{N})\|\phi_M\|_{L^2(-1,1)}.$$

Here no duality argument allows us to derive from the previous theorem improved approximation properties in negative norms. It is an open problem to derive such results.

The numerical analysis of problem (13) then continues by noticing that

$$\sum_{\mathrm{GL}} f v_N = \sum_{\mathrm{GL}} \mathcal{I}_N(f) v_N,$$

which is one of the ingredients that allows to prove (see [2]):

Theorem 9. *Assume that the solution u of (12) belongs to $H^r(\Omega)$, that the coefficients in A are very regular, and that the data f belongs to $H^\rho(\Omega)$. Then the solution u_N to (13) satisfies*

$$\|u - u_N\|_{H^1(\Omega)} \le c(N^{1-r}\|u\|_{H^r(\Omega)} + N^{-\rho}\|f\|_{H^\rho(\Omega)}).$$

The case where A is not so regular can be handled with the same type of arguments, but more technical tools are involved; we refer to [16] for more details. It is interesting also to note at this level that, taking into account non-homogeneous Dirichlet boundary condition is very simple thanks to the nice properties of the interpolation operator. Indeed, assume that the solution to our problem (12) has to satisfy (instead of zero Dirichlet boundary conditions) the following condition: $u_{|\partial\Omega} = g$ where g is a given function on the boundary of Ω. Then, naturally, for the approximation, we look for a polynomial u_N in $\mathbb{P}_N(\Omega)$ such that (12) holds, and in addition

$$u_{N|\partial\Omega} = \widetilde{\mathcal{I}}_N g,$$

where $\widetilde{\mathcal{I}}_N$ is the operator of interpolation defined edge by edge (respectively face by face) from i_N (resp. from $\mathcal{I}_N$). Since the interpolation operator is optimal both in L^2 and in H^1, it results by an argument of interpolation between Sobolev spaces that it is also optimal with respect to the $H^{1/2}(\partial\Omega)$–norm. This fractional order norm is the natural one for the treatment of the boundary terms. It has also to be stressed that neither the L^2–projection operator nor the H^1–projection operator allow such an optimality nor such ease in the implementation.

Next, associated with the Gauss quadrature formula, we can also define an interpolation operator, denoted as j_N and defined as follows: $\forall \phi \in \mathcal{C}^0([-1,1])$,

$$j_N(\phi) \in \mathbb{P}_{N+1}(-1,1) \text{ and } \forall i, 1 \le i \le N+1, \quad j_N(\phi)(\zeta_i) = \phi(\zeta_i).$$

The $L^2(-1,1)$–approximation properties of this second interpolation operator are also optimal. Unfortunately, in the H^1–norm it is not optimal; for instance it is readily checked that $j_N(L_{N+1} - L_{N-1}) = L_{N-1}$. Recalling that

$$\forall n, \quad L_{n+1} - L_{n-1} = -\frac{2n+1}{n(n+1)}(1-\zeta^2)L'_n,$$

it is then easily proven that $\|L_{N+1} - L_{N-1}\|_{H^1(-1,1)}$ scales like $\mathcal{O}(N^{1/2})$ while $\|L_{N-1}\|_{H^1(-1,1)}$ scales like $\mathcal{O}(N)$; j_N is thus not stable in the H^1 norm.

For similar reasons, the interpolation operator i_N on the Gauss-Lobatto nodes does not have optimal approximation properties in the $H^2(-1,1)$–norm. In order to achieve such a property, we have to refer to generalized Gauss-Lobatto rules as is done e.g.in [1].

§4. An "Ideal" Operator

At this stage there is no operator from $L^2(-1,1)$ onto the set of polynomials that has optimal approximation properties and is stable both in the L^2 and the H^1 norm. Such an operator is useful, as will be explained below, in the analysis of the Stokes problem. In order to define this "ideal" operator, we fix a positive real number λ and a cut-off function χ of class $\mathcal{C}^1$ on $\mathbb{R}^+$ such that χ is equal to 1 on $[0, 1-\lambda]$, decreases from 1 to 0 on $[1-\lambda, 1]$ and vanishes on $[1,\infty]$. Next, with each positive integer N, we associate as in [18] an operator π_N^χ with values in $\mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$ as follows: since each function ϕ in $H_0^1(-1,1)$ can be written as $\phi = \sum_{n=1}^\infty \widehat{\phi}^n(L_{n+1} - L_{n-1})$, we set $\pi_N^\chi \phi = \sum_{n=1}^\infty \chi(\frac{n}{N})\widehat{\phi}^n(L_{n+1} - L_{n-1})$. Note that the sum above is finite since χ has a bounded support. It is proven in [4] that this operator is stable both in the H_0^1 and the L^2 norms:

Theorem 10. *There exists a constant c, independent of N such that for any function* $\phi \in H_0^1(-1,1)$,

$$\|\pi_N^\chi \phi\|_{L^2(-1,1)} \leq c\|\phi\|_{L^2(-1,1)}, \quad \|\pi_N^\chi \phi\|_{H^1(-1,1)} \leq \|\phi\|_{H^1(-1,1)}. \tag{16}$$

It is an easy matter to verify that the operator π_N^χ leaves invariant all polynomials of $\mathbb{P}_{\lambda N}(-1,1) \cap H_0^1(-1,1)$. The previous stability and the best fit estimates (9), (10) imply

Theorem 11. *For any positive real number r and any real number* $0 \leq s \leq r$, *there exists a constant* $c > 0$, *depending only on r and s such that, for any function* $\phi \in H_0^r(-1,1)$ *if* $r \leq 1$ *and any function* $\phi \in H_0^1(-1,1) \cap H^r(-1,1)$ *if* $r \geq 1$,

$$\|\phi - \pi_N^\chi \phi\|_{H^s(-1,1)} \leq cN^{s-r}\|\phi\|_{H^r(-1,1)}.$$

As an application of the previous result, we can consider the problem of finding compatible spaces for the approximation of the Stokes equation. Under variational formulation, this problem consists in finding a pair $(\boldsymbol{u}, p)$ in $(H_0^1(\Omega))^d \times L_0^2(\Omega)$ of velocity and pressure such that

$$\int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v} - \int_\Omega p\mathrm{div}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\boldsymbol{v}, \quad \forall \boldsymbol{v} \in (H_0^1(\Omega))^d, \tag{17}$$

$$\int_\Omega q\mathrm{div}\boldsymbol{u} = 0, \quad \forall q \in L_0^2(\Omega), \tag{18}$$

where $L_0^2(\Omega)$ is the set of L^2 functions with zero average. It is well understood now that the spectral approximation of the Stokes problem based on polynomials of the *same* degree leads to instabilities. This is due to the fact that the pressure space is too rich in comparison to the velocity space. Indeed, there exist polynomials q_N in $\mathbb{P}_N((-1,1)^d)$ such that $\int_\Omega q_N \mathrm{div}\boldsymbol{v}_N = 0$ for all $\boldsymbol{v}_N$ in $(\mathbb{P}_N((-1,1)^d) \cap H_0^1((-1,1)^d))^d$ (e.g. $q_N(x,y,z) = L_N(x)L_N(y)L_N(z)$). Of course such polynomials (called spurious modes) prevent the discrete problem from being well-posed since it prevents the definition of a unique pressure. The cure is well known, and consists in depleting the pressure space for a given velocity space. In [14] the pair $(\mathbb{P}_N((-1,1)^d) \cap H_0^1((-1,1)^d))^d \times \mathbb{P}_{N-2}((-1,1)^d) \cap L_0^2((-1,1)^d)$ has been proposed, and gets rid of the spurious modes. It is known as the $\mathbb{P}_N \times \mathbb{P}_{N-2}$–method. Actually, what is looked for is a pair $X_N \times M_N$ approximating $(H_0^1((-1,1)^d))^d \times L_0^2((-1,1)^d)$ well and such that not only $\forall q_N \in M_N, \exists \boldsymbol{v}_N, \ \int_\Omega p\mathrm{div}\boldsymbol{v}_N \neq 0$, but more precisely, in order to get a stable method, we require that

$$\forall q_N \in M_N, \exists \boldsymbol{v}_N, \quad \int_\Omega p\mathrm{div}\boldsymbol{v}_N \geq \beta\|\boldsymbol{v}_N\|_{H^1((-1,1)^d)}\|q_N\|_{L^2((-1,1)^d)},$$

where β is known as the constant of the inf-sup condition. The behaviour of β for the $\mathbb{P}_N \times \mathbb{P}_{N-2}$–method scales as $\mathcal{O}(N^{-\frac{d-1}{2}})$ (see [2]), and it has been a long standing question whether there is a uniformly stable spectral approximation of the Stokes problem. It has to be said that the nonuniform behaviour of the inf-sup constant pollutes the accuracy of the pressure, but also pollutes the convergence properties of some classical solvers for the Stokes problem (see [13]). The "ideal" operator introduced above allows us to prove that a compatible choice is the $\mathbb{P}_N \times \mathbb{P}_{\lambda N}$-method that proposes, for the same choice of velocity space, $\mathbb{P}_{\lambda N}((-1,1)^d) \cap L^2_0((-1,1)^d)$ to be the pressure space. The following result is due to Bernardi and Maday [4]:

Theorem 12. *For any real number λ, $0 < \lambda < 1$, there exists a positive constant β independent of N such that, for any integer $N \geq 2/(1-\lambda)$ and any $q_N \in \mathbb{P}_{\lambda N}((-1,1)^d) \cap L^2_0((-1,1)^d)$,*

$$\sup_{\boldsymbol{v}_N \in (\mathbb{P}_N((-1,1)^d) \cap H^1_0((-1,1)^d))^d} \frac{\int_\Omega p \, div \boldsymbol{v}_N}{\|\boldsymbol{v}_N\|_{H^1((-1,1)^d)}} \geq \beta \|q_N\|_{L^2((-1,1)^d)}.$$

Proof: Let q_N be any polynomial in $\mathbb{P}_{\lambda N}((-1,1)^d) \cap L^2_0((-1,1)^d)$. It is a standard matter (see e.g. Corollary 2.4 in [8]) that, to q_N, can be associated a (continuous) element $\boldsymbol{v}$ in $[H^1_0((-1,1)^d)]^d$ such that

$$\text{div}\boldsymbol{v} = q_N \text{ and } \|\boldsymbol{v}\|_{H^1((-1,1)^d)} \leq c \|q_N\|_{L^2((-1,1)^d)}.$$

The problem is that $\boldsymbol{v}$ is not a polynomial. We define $\boldsymbol{v}_N = \pi^\chi_N \otimes \pi^\chi_N \boldsymbol{v}$ in 2D and $\boldsymbol{v}_N = \pi^\chi_N \otimes \pi^\chi_N \otimes \pi^\chi_N \boldsymbol{v}$ in 3D for which we derive thanks to (13) that

$$\|\boldsymbol{v}_N\|_{H^1((-1,1)^d)} \leq c \|q_N\|_{L^2((-1,1)^d)}.$$

Due to the fact that π^χ_N leaves invariant all polynomials of $\mathbb{P}_{\lambda N}(-1,1) \cap H^1_0(-1,1)$, we deduce that $\int_\Omega q_N \text{div}(\boldsymbol{v}_N - \boldsymbol{v}) = 0$, and thus

$$\int_\Omega q_N \text{div}\boldsymbol{v}_N = \int_\Omega q_N \text{div}\boldsymbol{v} = \int_\Omega q_N^2,$$

which concludes the proof with $\beta = \frac{1}{c}$. □

§5. Extension to Domain Decompositions

In the spectral method history, the need to tackle more general domains was recognized early. In this direction, Patera has proposed in [19] the spectral element method that combines the accuracy of the spectral method with the flexibility of the domain decomposition methods. The idea is to introduce a partition of the domain Ω as a union of nonoverlapping subdomains:

$$\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}^k, \quad \Omega^k \cap \Omega^\ell = \emptyset.$$

In addition, we assume that each subdomain Ω^k is associated with a regular one-to-one mapping $\mathcal{F}^k$ that maps the brick $(-1,1)^d$ onto Ω^k and, for the time being at least, we make the following assumptions:

Assumption 1.

$$\overline{\Omega}^k \cap \overline{\Omega}^\ell = \begin{cases} \text{an entire common face (in 3D),} & \text{or} \\ \text{an entire common edge,} & \text{or} \\ \text{a common vertex,} & \text{or} \\ \emptyset & \end{cases}$$

Assumption 2. The two parametrizations of the previous intersection $\overline{\Omega}^k \cap \overline{\Omega}^\ell$, resulting from $\mathcal{F}^k$ and $\mathcal{F}^\ell$, coincide.

This allows us to define the discrete space

$$X_N = \{v_N \in H_0^1(\Omega), v_{N|\Omega^k} \circ \mathcal{F}^k \in \mathbb{P}_N((-1,1)^d)\}$$

and the discrete associated problem (2) (or its implementable version involving the Gauss-Lobatto quadrature rule over each Ω^k as in (13)).

The main ingredient that allows us to prove that the previous scheme is again optimal lies in the definition of a element in X_N that approximates well a given regular function u. This is done easily by considering the element v_N, defined locally over each subdomain as $v_{N|\Omega^k} \circ \mathcal{F}^k = \mathcal{I}_N[u_{|\Omega^k} \circ \mathcal{F}^k]$. It results from Assumptions 1 and 2 that v_N is actually continuous and vanishes over $\partial\Omega$. From (15) it is an optimal approximation of u in the sense that

$$\|u - v_N\|_{H^1(\Omega)} \leq cN^{1-r}\|u\|_{H^r(\Omega)}. \tag{19}$$

The best fit in $H^1(\Omega)$ is certainly as good as the proposed v_N, and the spectral element method can be proven to be an optimal approximation. We have only sketched the numerical analysis of this approximation, since the main purpose of this paper is to discuss projection operators. It is fundamental to have used here the interpolation operator to construct v_N, since it provides a globally continuous function. As an example, the use of the H^1–projection operator would not have given rise to a continuous function since, for a given function ϕ over the brick $(-1,1)^d$, the value of $\Pi_N^{1,0}(\phi)$ over any face depends not only on the value of ϕ on the given face, but depends on ϕ inside the whole domain.

We want to end this section by giving some hints on the "mortar spectral element method" due to Bernardi, Maday and Patera, that allows to relax assumptions 1 and 2 (and even, more generally, allows to combine spectral methods on some subdomains with different finite element methods on others see [5]). Due to lack of space, but also in order to better understand the main feature of the projection operators that is at the basis of the method, we shall consider a simple two dimensional domain $\Omega = (-1,2) \times (-1,1)$ decomposed into 3 subdomains $\Omega^1 = (-1,1) \times (-1,1)$, $\Omega^2 = (1,2) \times (-1,0)$ and $\Omega^3 = (1,2) \times (0,1)$. This decomposition violates assumption 1 since the intersection $\overline{\Omega}^1 \cap \overline{\Omega}^2$ is not a common *whole* edge. We want nevertheless to propose a discrete method that will allow to provide an optimal approximation of the solution u of (12) (with A =Id for the sake of simplicity). The discrete space X_N^* that we propose is imbedded in

$$Y_N = \{v_N \in L^2(\Omega), v_{N|\Omega^k} \in \mathbb{P}_N, v_{N|\partial\Omega} = 0, v_{N|\Omega^2} = v_{N|\Omega^3} \text{ over } \overline{\Omega}^2 \cap \overline{\Omega}^3\}$$

but it is readily checked that imposing continuity at the level of the interface $x = 1$ will rigidify the approximation and, in the general case, will spoil the accuracy of the method. In order to relax this continuity condition (remind that it is inherited from the requirement that $X_N^* \subset X$), we resort to nonconforming approximations. We shall replace the continuity condition by requiring that, over the interface $x = 1$, we impose for each element in Y_N

$$\int_{-1}^{1} [v^-(1,y) - v^+(1,y)]\psi_N(y)dy = 0, \quad \forall \psi_N \in \mathbb{P}_{N-2}(-1,1), \tag{20}$$

where $v^- = v_{|\Omega^1}$ and

$$v^+ = \begin{cases} v_{|\Omega^2} & \text{for } (x,y) \in \Omega^2, \\ v_{|\Omega^3} & \text{for } (x,y) \in \Omega^3. \end{cases}$$

Since $v^-(1,y)$ has to vanish for $y = \pm 1$ (due to the homogeneous boundary conditions), it is entirely defined by the $N-1$ conditions in (20); in particular choosing ψ in $\mathbb{P}_N(-1,1)$ would be much too stringent. The elements of Y_N that satisfy (20) constitute the space X_N^* of approximation. The method is then: find $u_N^* \in X_N^*$ such that

$$a_N(u_N^*, v_N) \equiv \sum_{k=1}^{K} \int_{\Omega^k} \nabla u_N^* \nabla v_N = \sum_{k=1}^{K} \int_{\Omega^k} f v_N, \quad \forall v_N \in X_N^*. \tag{21}$$

Since X_N^* is no longer a subspace of X, the ellipticity of the bilinear form of this problem is not straightforward. Nevertheless, it is true (and here it is particularly obvious since $\partial\Omega^k \cap \partial\Omega \neq \emptyset$). This argument allows us to check that there exists a unique solution u_N^* to (21). In order to derive the error bound we proceed as follows: for any $w_N \in X_N^*$,

$$\begin{aligned}
\alpha \|u_N^* - w_N\|_*^2 &\leq a_N(u_N^* - w_N, u_N^* - w_N) \\
&= \sum_{k=1}^{K} \int_{\Omega^k} f(u_N^* - w_N) - \sum_{k=1}^{K} \int_{\Omega^k} \nabla w_N \nabla (u_N^* - w_N) \\
&= \sum_{k=1}^{K} \int_{\Omega^k} -\Delta u(u_N^* - w_N) - \sum_{k=1}^{K} \int_{\Omega^k} \nabla w_N \nabla (u_N^* - w_N) \\
&= \sum_{k=1}^{K} \int_{\Omega^k} \nabla u \nabla (u_N^* - w_N) - \sum_{k=1}^{K} \int_{\Omega^k} \nabla w_N \nabla (u_N^* - w_N) \\
&\quad - \int_{x=1} \frac{\partial u}{\partial x}[(u_N^* - w_N)^- - (u_N^* - w_N)^+],
\end{aligned}$$

so that, from (20) we derive that for any $\psi \in \mathbb{P}_{N-2}(-1,1)$

$$\begin{aligned}
\alpha \|u_N^* - w_N\|_*^2 \leq \sum_{k=1}^{K} \int_{\Omega^k} \nabla (u - w_N) \nabla (u_N^* - w_N) \\
- \int_{x=1} [\frac{\partial u}{\partial x} - \psi][(u_N^* - w_N)^- - (u_N^* - w_N)^+].
\end{aligned}$$

It follows from the previous inequality that

$$\alpha\|u_N^* - w_N\|_* \le c\|u - w_N\|_* + \sup_{v_N \in X_N^*} \frac{\int_{x=1}[\frac{\partial u}{\partial x} - \psi][v_N^- - v_N^+]dy}{\|v_N\|_*}. \tag{22}$$

By choosing ψ equal to $\pi_{N-2}\frac{\partial u}{\partial x}$, it results that

$$\sup_{v_N \in X_N^*} \frac{\int_{x=1}[\frac{\partial u}{\partial x} - \psi][v_N^- - v_N^+]dy}{\|v_N\|_*} \le cN^{-r}\|u\|_{H^r(\Omega)}.$$

It remains to choose a good approximation w_N of u in X_N^* to take into account the first term on the right hand side of (22). This is done by noticing that, for any $\phi \in H_0^1(-1,1)$, the element ϕ_N of $\mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$ that satisfies

$$\int_{-1}^{1} [\phi_N - \phi]\psi_N(y)dy = 0, \quad \forall \psi_N \in \mathbb{P}_{N-2}(-1,1),$$

is nothing other than $\phi_N = \pi_N^{1,0}(\phi)$. Indeed, we remark that, for any $\chi_N \in \mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$, then $\chi_N'' \in \mathbb{P}_{N-2}(-1,1)$, thus

$$\int_{-1}^{1} [\phi_N - \phi]\chi_N''(y)dy = -\int_{-1}^{1} [\phi_N - \phi]'\chi_N'(y)dy.$$

The choice of a good element w_N is done as follows. We first set $\tilde{w}_{N|\Omega^k} = I_N(u_{N|\Omega^k})$ that is an element of Y_N. We then set $w_{N|\Omega^k} = \tilde{w}_{N|\Omega^k}$ for $k = 2, 3$, and build $w_{N|\Omega^1}$ by adding to $\tilde{w}_N^-$ the correction $\pi_N^{1,0}(\tilde{w}_N^+ - \tilde{w}_N^-)(y)\frac{(1+x)L_N'(x)}{2L_N'(1)}$ so that it satisfies (20). Due to the optimal approximation properties of the operator $\pi_N^{1,0}$ both in the L^2 and in the H^1–norms, we deduce that the mortar spectral approximation (21) is optimal in the sense that (19) still holds.

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